Vehicle Lateral Dynamics Control

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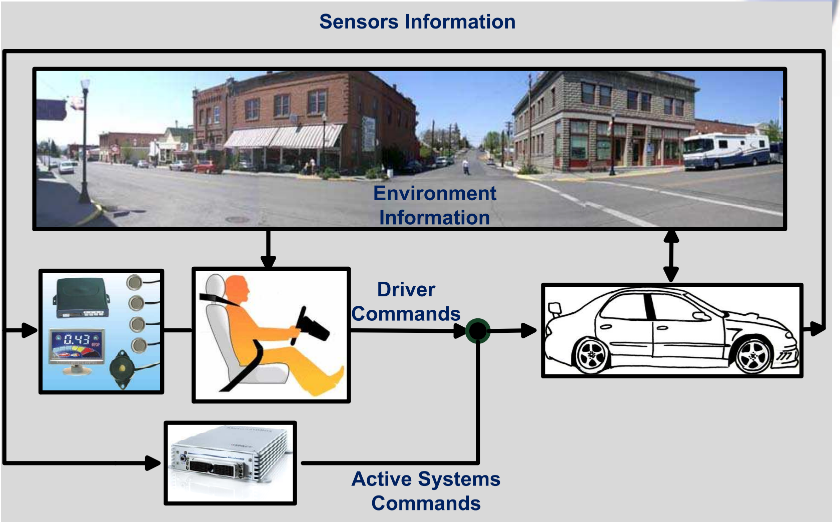
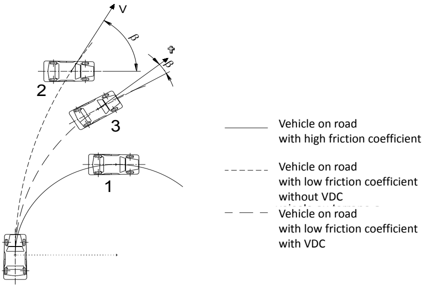
Outline

Closed Loop System Environment-Driver-Vehicle



Outline

The Roles of VDC



- If the friction coefficient is small or the vehicle speed is too high, then the vehicle may be unable to track the nominal trajectory and may follow the trajectory of larger radius.
- Then, one of the goals of the lateral control system is the yaw velocity of the vehicle to track as much as possible the nominal motion expected by the driver, as shown by the middle curve in Figure.

SAE Definition

The Society of Automotive Engineers (SAE) defined ESC as a system that has all of the following attributes:

a) ESC increases vehicle directional stability by applying and adjusting the vehicle brakes individually to induce correcting yaw torques to the vehicle.

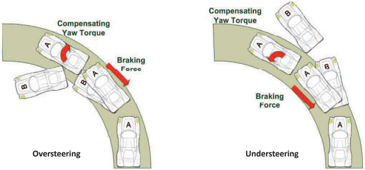
b) ESC is a computer-controlled system, which uses a closed-loop algorithm to limit understeer and oversteer of the vehicle when appropriate.

c) ESC has means to determine vehicle yaw rate and to estimate its sideslip.

d) ESC has means to monitor driver steering input.

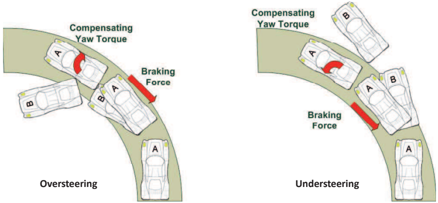
e) ESC is operational over the full speed range of the vehicle (except below a low speed threshold where loss of control is unlikely).

Under/Oversteering



Oversteering. In Figure to the left, the vehicle has entered a left curve that is extreme for the speed it is traveling. The rear of the vehicle begins to slide which would lead to a non-ESC vehicle turning sideways (or spinning out) unless the driver expertly countersteers. In a vehicle equipped with ESC, the system immediately detects that the vehicles heading is changing more quickly than appropriate for the drivers intended path (the yaw rate and side slip angle are too high). It momentarily applies the right front brake to turn the heading of the vehicle back to the correct path.

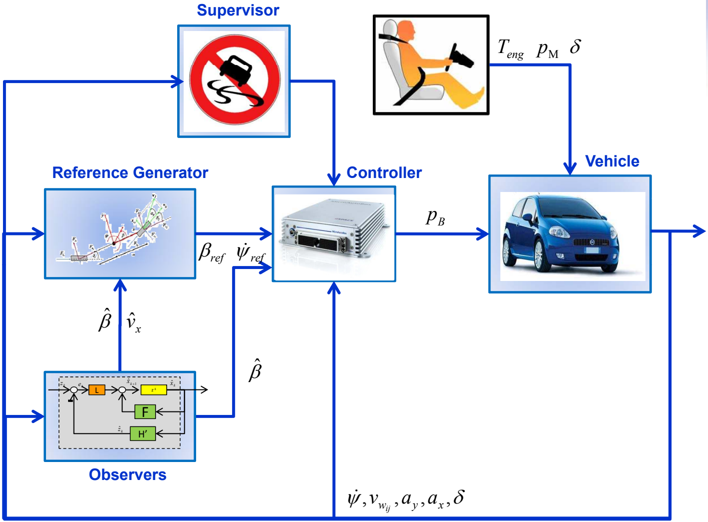
Under/Oversteering



Understeering. Figure at the right shows a similar situation faced by a vehicle whose response as it nears the limits of road traction is first sliding at the front (plowing out or understeering) rather than oversteering. In this vehicle, ESC rapidly detects that the vehicles heading is changing less quickly than appropriate for the drivers intended path (the yaw rate is too low). It momentarily applies the left rear brake to turn the heading of the vehicle back to the correct path.

Outline

General Control Scheme

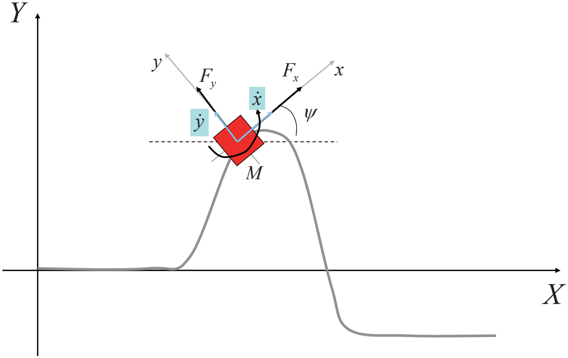


General Control Scheme

- The main actor is the driver who can generate or variate the car dynamics changing the accelerator pedal position, the steering wheel angle and braking pedal position.
- Under the driver's input, the vehicle moves and gives information about its dynamics through its sensors that are:
  1. Steering wheel angle sensor: determines the driver's intended rotation; i.e. where the driver wants to steer. This kind of sensor is often based on AMR (Anisotropic MagnetoResistance) elements.
  2. Yaw rate sensor: measures the yaw rate of the car; i.e how much the car is actually turning.
  3. Lateral and longitudinal acceleration sensor: often based on the Hall effect, it measures the lateral and longitudinal acceleration of the vehicle.
  4. Wheel speed sensor: measures the wheel speed (e.g., via a phonic wheel)

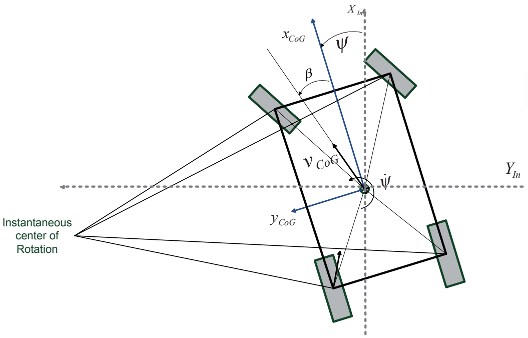
Outline

A simplified vehicle model



$$
\begin{aligned}
ma_x &= F_x, & \text{(1a)}\\
ma_y &= F_y, & \text{(1b)}\\
I\ddot{\psi} &= M. & \text{(1c)}
\end{aligned}
$$

Coordinate Transformation



The CoG velocity of vehicle in the Inertial Frame can be translated in the body-frame using the following trigonometric transformation

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = v_{CoG} \begin{bmatrix} \cos(\beta + \psi) \\ \sin(\beta + \psi) \end{bmatrix} \tag{2}$$

Three-State Four-Wheel Model

By differentiation the equation (2), the accelerations in (1) are obtained:

$$\begin{bmatrix} \ddot{X} \\ \ddot{Y} \end{bmatrix} = v_{CoG}\left(\dot{\beta}+\dot{\psi}\right)\begin{bmatrix} -\sin\left(\beta+\psi\right) \\ \cos\left(\beta+\psi\right) \end{bmatrix} + \dot{v}_{CoG}\begin{bmatrix} \cos\left(\beta+\psi\right) \\ \sin\left(\beta+\psi\right) \end{bmatrix} \quad (3)$$

These accelerations have to be reported in the body frame. For this reason, it is necessary to introduce the *rotation matrix* that permits, through a compact matrix notation, to make a rotation around one axis (in our case, the axis is the z one), then,

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} \ddot{X} \\ \ddot{Y} \end{bmatrix}. \quad (4)$$

Three-State Four-Wheel Model

Finally, the kinematic expressions of body frame accelerations are

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = v_{CoG}\left(\dot{\beta}+\dot{\psi}\right)\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} + \dot{v}_{CoG}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}. \tag{5}$$

Substituting the equation (5) in equations (1b) and (1a) we obtain

$$v_{CoG}\left(\dot{\beta}+\dot{\psi}\right)\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} + \dot{v}_{CoG}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = \frac{1}{m}\begin{bmatrix} F_{x_{fl}} + F_{x_{fr}} + F_{x_{rl}} + F_{x_{rr}} \\ F_{y_{fl}} + F_{y_{fr}} + F_{y_{rl}} + F_{y_{rr}} \end{bmatrix} \tag{6a}$$

Considering as state variables the velocity of vehicle v_{CoG} and the side slip angle β, from (6) it is possible to obtain the following two state equations,

$$\dot{v}_{CoG} = \frac{1}{m\cos\beta}\sum F_{x_{ij}} + v_{CoG}\left(\dot{\beta}+\dot{\psi}\right)\tan\beta \tag{7a}$$

$$\dot{\beta} = \frac{1}{mv_{CoG}\cos\beta}\left(\sum F_{y_{ij}} - m\dot{v}_{CoG}\sin\beta\right) - \dot{\psi} \tag{7b}$$

Three-State Four-Wheel Model

and deleting the mutual dependency between $\dot{v_{CoG}}$ and $\dot{\beta}$ it is possible to rewrite the equations (7) as

$$\dot{v}_{CoG} = \frac{\cos\beta}{m}\sum F_{x_{ij}} + \frac{1}{m}\sum F_{y_{ij}}\sin\beta \tag{8a}$$

$$\dot{\beta} = \frac{\cos\beta}{v_{CoG}m}\sum F_{y_{ij}} - \frac{\sin\beta}{v_{CoG}m}\sum F_{x_{ij}} - \dot{\psi} \tag{8b}$$

Furthermore, the third state equation is provided by rotation yaw moment as

$$I\ddot{\psi} = \left(F_{y_{fr}} + F_{y_{fl}}\right)a - \left(F_{y_{rr}} + F_{y_{rl}}\right)b + \left(-F_{x_{fr}} + F_{x_{fl}} - F_{x_{rl}} + F_{x_{rr}}\right)c \tag{9}$$

where a is longitudinal distance from CoG to the front axle, b is longitudinal distance from CoG to the rear axle, and c is the semi-track width.

Three-State Four-Wheel Model: Tire Forces

The lateral and longitudinal forces are derived from the lateral and longitudinal tire force, respectively, in the following way

$$F_{y_{\star,\bullet}} = F_{l_{\star,\bullet}} \sin\delta_\star + F_{c_{\star,\bullet}} \cos\delta_\star \tag{10a}$$

$$F_{x_{\star,\bullet}} = F_{l_{\star,\bullet}} \cos\delta_\star + F_{c_{\star,\bullet}} \sin\delta_\star. \tag{10b}$$

$F_{c_{\star,\bullet}}$ and $F_{l_{\star,\bullet}}$ are complex functions of several parameters. A possible dependency can be describes as

$$F_{c_{\star,\bullet}} = f_c(\alpha_{\star,\bullet}, s_{\star,\bullet}, \mu_{\star,\bullet}, F_{z_{\star,\bullet}}), \tag{11a}$$

$$F_{l_{\star,\bullet}} = f_l(\alpha_{\star,\bullet}, s_{\star,\bullet}, \mu_{\star,\bullet}, F_{z_{\star,\bullet}}), \tag{11b}$$

where $\alpha_{\star,\bullet}$ are the tire slip angles, $s_{\star,\bullet}$ are the slip ratios, $\mu_{\star,\bullet}$ are the road friction coefficients and $F_{z_{\star,\bullet}}$ are the tires normal forces.

Three-State Four-Wheel Model

Using the equations (8)-(11) the nonlinear vehicle dynamics can be described by the following compact differential equation:

$$\dot{\xi}(t) = f^{4w}_{\mu(t)}(\xi(t), u(t)), \tag{12}$$

where the state and input vectors are $\xi = [\ v_{CoG},\ \beta,\ \dot{\psi}]$ and $u = [\delta_f,\ F_{l_{f,l}},\ F_{l_{f,r}},\ F_{l_{r,l}},\ F_{l_{r,r}}]$, respectively, and $\mu(t) = [\mu_{f,l}(t),\ \mu_{f,r}(t),\ \mu_{r,l}(t),\ \mu_{r,r}(t)]$.

Two-State Four-Wheel model

Remark

Simplification: *The velocity of CoG v_{CoG} is constant and known.*

From the Simplification above the equation (8a) can be discarded, then the system (13) is reduced to equation (8b) and (9). In state space form, the reduced nonlinear two-track model, for a given friction coefficient, can be written as

$$\dot{\xi}(t) = f^{4w}_{\mathrm{2states}}(\xi(t), u(t)), \tag{13}$$

where the state and input vector are $\xi = [\beta, \dot{\psi}]$ and $u = [\delta_f, F_{l_{fl}}, F_{l_{fr}}, F_{l_{rl}}, F_{l_{rr}}]$. The subscript '2states' reminds that the model (13) is a four-wheel model with only two state variables.

Outline

Nominal Trajectory

The ESC system generates at each time step the reference trajectory from the input of the driver

- car longitudinal velocity
- steering wheel angle

This "nominal trajectory" is characterized by two angular quantities, yaw rate $\dot{\psi}_{\mathrm{ref}}$ and β_{ref}, on the body frame rather than the inertial frame because it depends on the driver's intentions so that it is impossible to draw an absolute trajectory.

Yaw Rate Reference

To compute the reference yaw rate signal we use the relation between a given steering angle and the corresponding yaw rate at steady state for a bicycle model, sometimes called Ackermann yaw rate,

$$\dot{\psi}_{\mathrm{Ack}} := \frac{v_x}{l\left(1 + v_x^2/v_{ch}^2\right)}\delta, \tag{14}$$

where $l = a + b$.

Yaw Rate Reference

The Ackermann yaw rate is the steady-state yaw rate corresponding to the given steering wheel angle of a linearized bicycle model at a determined longitudinal velocity.
It needs to be smoothed and to this aim we employ a unit gain low pass filter whose poles are obtained by the linearization around $\beta = 0$ and $\dot{\psi} = 0$ of the bicycle model:

$$W\left(p\right) := \frac{a}{p^2 + bp + a}, \tag{15}$$

with

$$\begin{aligned} a(v_x) =& \frac{c_F c_R l^2 + m v^2 \left(c_R l_b - c_F l_a\right)}{J_z m v_x}, \\ b(v_x) =& \frac{\left(J_z + m l_a^2\right) c_F + \left(J_z + m l_b^2\right) c_R}{J_z m v_x}. \end{aligned}$$

Yaw Rate Reference

Further, to take into account the saturation of tire lateral forces not described by the linearized model, some bounds are introduced, typically referring to oversteering and understeering behaviors. For an oversteering car if we consider the derivative of side slip angle $\dot{\beta}$ approximately zero the equation (7) becomes

$$\dot{\psi} \approx \frac{1}{v_{CoG} \cos\beta} \left(\sum \frac{F_{y_{ij}}}{m} - \dot{v}_{CoG} \sin\beta \right), \tag{16}$$

where

$$a_Y = \sum \frac{F_{y_{ij}}}{m}. \tag{17}$$

Yaw Rate Reference

From equation (16) it is possible to compute

$$\dot{\psi}_{\max} = \frac{1}{v_{CoG} \cos\beta} \left(a_{Y_{\max}} - \dot{v}_{CoG} \sin\beta \right). \tag{18}$$

The reference yaw rate ψ_{ref},in the case of oversteering, has to be inside this bound,

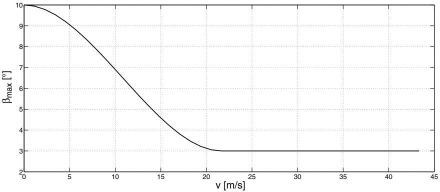
$$\dot{\psi}_{\mathrm{ref}} = \begin{cases} \dot{\psi}_{\mathrm{Ack}}, & |\dot{\psi}| \leq |\dot{\psi}_{\max}| \\ \pm\dot{\psi}_{\max}, & \text{otherwise} \end{cases} \tag{19}$$

Yaw Rate Reference

In the case of understeering, the vehicle side slip angle β and the yaw rate $\dot{\psi}$ are below their maximum allowable values. The driver tries to maintain the vehicle on the desired course by increasing the steering angle. If the tire slip angle α and, therefore the lateral wheel slip s_c, become too large, the lateral friction coefficient exceeds the maximum. The vehicle would then leave the set course. It is possible to use the rear tire slip angle α_r as a reference to determine when the front tire side slip angle α_f reaches a critical value. A critical ratio is $\alpha_f/\alpha_r = 1.5$. Then, for an understeering car, the reference yaw rate $\dot{\psi}_{\text{ref}}$ must satisfy

$$\dot{\psi}_{\text{ref}} = \begin{cases} \pm\dot{\psi}_{\max}, & \alpha_f/\alpha_r \geq 1.5 \\ \dot{\psi}_{\text{Ack}}, & \text{otherwise} \end{cases} \tag{20}$$

Side Slip Angle Reference



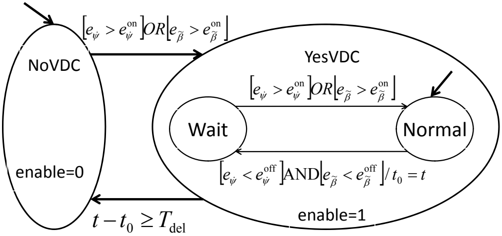
We require $\widetilde{\beta}$ to be limited in the interval $[-\beta_{\max}, \beta_{\max}]$ where

$$\beta_{\max} := \begin{cases} 2\frac{k_1-k_2}{v_{ch}^3} v_x^3 - 3\frac{k_1-k_2}{v_{ch}^2} v_x^2 + k_1 & \text{if } v_x < v_{ch}, \\ k_2 & \text{if } v_x \geq v_{ch}. \end{cases} \tag{21}$$

Reasonable values for parameters k_1 and k_2 are $10\pi/180$ and $3\pi/180$ respectively. When $\beta(t) \in [-\beta_{\max}, \beta_{\max}]$ the side slip reference is itself; when $\beta(t) > \beta_{\max}$ (resp. $\beta(t) < -\beta_{\max}$) we have $\beta_{\text{ref}} = \beta_{\max}$ (resp. $\beta_{\text{ref}} = -\beta_{\max}$).

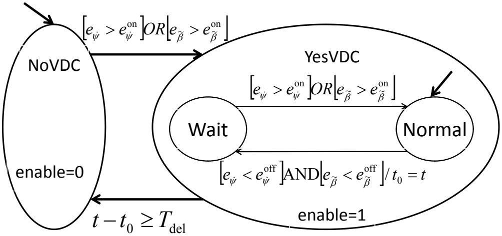
Outline

Supervisor



- The lateral dynamics controller is not always ON so as to leave the driver free during normal driving.
- It is turned on by a Supervisor that evaluates the error $e_{\dot\psi}$ between the actual yaw rate $\dot\psi$ and the reference yaw rate $\dot\psi_{\text{ref}}$ and the error e_β between the estimated vehicle side slip angle $\hat\beta$ and the reference vehicle side slip angle $\hat\beta_{\text{ref}}$.

Supervisor



- The Supervisor activates when either the error on yaw rate $e_{\dot\psi}$ or the error on side slip angle e_β exceed certain respective activations thresholds.
- The controller is deactivated when both $e_{\dot\psi}$ and $e_{\tilde\beta}$ are within those thresholds for a period T_{del}.

Yaw Rate Thresholds

- The thresholds on yaw rate error depends on the vehicle speed.
- Since the vehicle responds to the steering wheel angle in different manners as the vehicle speed changes, we chose to shape the yaw rate error activation threshold $e_{\dot{\psi}}^{\text{on}}$ and the deactivation threshold $e_{\dot{\psi}}^{\text{off}}$ as the following

$$e_{\dot{\psi}}^{\text{on}}(v_x) = \frac{2v_x/v_{ch}}{\left(1 + v_x^2/v_{ch}^2\right)} e_{\dot{\psi}}^{\text{ON}}, \tag{22a}$$

$$e_{\dot{\psi}}^{\text{off}}(v_x) = \xi e_{\dot{\psi}}^{\text{on}}(v_x) : \tag{22b}$$

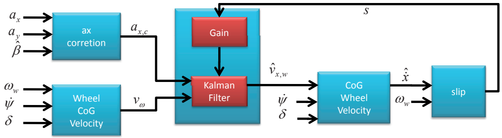
where $e_{\dot{\psi}}^{\text{ON}} > 0$ is a calibration parameter. Equation (22a) is obtained from (14) by setting $\delta = 1$ and dividing by $v_{ch}/(2l)$ (which is the maximum yaw rate for $\delta = 1$ attained at $v_x = v_{ch}$); ξ is a calibration parameter with $\xi \in (0, 1)$, tipically $\xi = 0.75$.

Outline

Observers

- The vehicle side slip angle β cannot be measured and has to estimated.
- To this aim the longitudinal velocity of CoG of vehicle is needed, another information not provided by sensors.
- Then, it is necessary to have
  - an estimation of longitudinal velocity of vehicle $\dot{x}$
  - an estimation of vehicle side slip β

Observers



The functional structure of vehicle longitudinal velocity estimator is showed in figure. Its inputs are:

1. angular velocity of each wheel: ω_{fl}, ω_{fr}, ω_{rl}, $\omega_{rr}[\mathrm{rad/s}]$;
2. vehicle longitudinal acceleration: $a_x[\mathrm{m/s^2}]$;
3. vehicle lateral acceleration: $a_y[\mathrm{m/s^2}]$;
4. the steering wheel angle imposed by driver: $SWA[\mathrm{rad}]$ $(\delta = SWA/R_{steer}[\mathrm{rad}])$;
5. the vehicle yaw rate: $\dot{\psi}[\mathrm{rad/s}]$;
6. the estimated vehicle side slip angle $\tilde{\beta}[\mathrm{rad}]$.

Estimation of CoG velocity

The velocity of each CoG of wheel can be expressed as

$$v_{w_{fl}} = \frac{\omega_{fl} r - \dot{\psi}\left(c\cos\delta - a\sin\delta\right)}{\cos\left(\delta - \beta\right)} \tag{23a}$$

$$v_{w_{fr}} = \frac{\omega_{fr} r + \dot{\psi}\left(c\cos\delta - a\sin\delta\right)}{\cos\left(\delta - \beta\right)} \tag{23b}$$

$$v_{w_{rl}} = \frac{\omega_{rl} r + c\dot{\psi}}{\cos\beta} \tag{23c}$$

$$v_{w_{rr}} = \frac{\omega_{rr} r - c\dot{\psi}}{\cos\beta} \tag{23d}$$

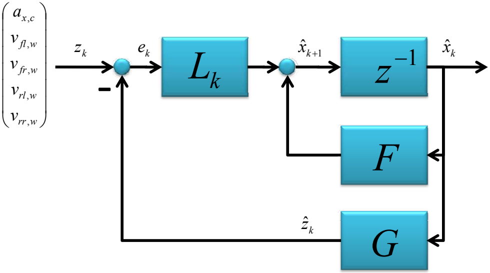
Kalman Filter

A Kalman filter, whose scheme is shown in figure, is used to estimate the vehicle longitudinal velocity v_x. It is designed on an augmented system whose output variables are the vehicle acceleration and the wheel longitudinal velocities computed as in equation (23)

$$\begin{pmatrix} \dot{a}_x \\ \dot{v}_x \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} a_x \\ v_x \end{pmatrix} + I_2 \sigma \tag{24a}$$

$$\begin{pmatrix} \hat{a}_x \\ \hat{v}_{w_{fl}} \\ \hat{v}_{w_{fr}} \\ \hat{v}_{w_{rl}} \\ \hat{v}_{w_{rr}} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a_x \\ v_x \end{pmatrix} + I_5 w \tag{24b}$$

Kalman Filter



The system (24) is discretized with a sampling time $T_s = 20$ ms

$$\hat{x}_{k+1} = F\hat{x}_k + G\sigma_k \tag{25}$$

$$\hat{z}_k = H'x_k + Iw_k \tag{26}$$

where $E[\sigma_k \sigma_k^T] = Q$ and $E[w_k w_k^T] = R$.

Kalman Filter

The equation of the Kalman filter are

$$x_{k|k} = x_{k|k-1} + L_k \left(z_k - H' x_{k|k-1}\right) \tag{27a}$$

$$x_{k+1|k} = F x_{k|k} \tag{27b}$$

where $z_k = \begin{bmatrix} a_x & \hat{v}_{w,FL} & \hat{v}_{w,FR} & \hat{v}_{w,RL} & \hat{v}_{w,RR} \end{bmatrix}'$ is the vector of measurements and the gain L_k is computed dynamically by

Kalman Filter

$$\Sigma_{k|k-1} = F\Sigma_{k-1|k-1}FH^T + Q_k \tag{28}$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1}H^T \left(H\Sigma_{k|k-1}H + R_k\right)^{-1} H\Sigma_{k|k-1} \tag{29}$$

$$L_k = \Sigma_{k|k-1}H^T \left(H\Sigma_{k|k-1}H + R_k\right)^{-1} \tag{30}$$

where by definition

$$\Sigma_{k|k-1} := E\left[\left(x - \hat{x}_{k|k-1}\right)\left(x - \hat{x}_{k|k-1}\right)^T\right] \tag{31}$$

$$\Sigma_{k|k} := E\left[\left(x - \hat{x}_{k|k}\right)\left(x - \hat{x}_{k|k}\right)^T\right]. \tag{32}$$

Typically the error covariance on velocity measurement is larger than that one on acceleration measurement which guarantees a fast convergence of the filter in the first time steps.

Kalman Filter

For the following time steps, the velocity measurements error covariance is a function of the slip ratio of each wheel, a sort of quality factor of the single measurement:

$$R = r \begin{bmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 - e^{\alpha \cdot \left| s_{L,FL} \right|} & 0 & 0 & 0 \\ 0 & 0 & 1 - e^{\alpha \cdot \left| s_{L,RL} \right|} & 0 & 0 \\ 0 & 0 & 0 & 1 - e^{\alpha \cdot \left| s_{L,FR} \right|} & 0 \\ 0 & 0 & 0 & 0 & 1 - e^{\alpha \cdot \left| s_{L,RR} \right|} \end{bmatrix}$$
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

Side Slip Angle Estimation

From (7b) the side slip angle estimate $\tilde{\beta}$ is given by

$$\dot{\tilde{\beta}} = -\frac{\dot{v}}{v_x} \sin \tilde{\beta} + \frac{a_y}{v_x} - \dot{\psi}, \tag{34}$$

where in turn v and v_x are estimated according to previous slides. By practical experience we noticed that (34) has a problem of error propagation, and thus, a dynamical reset $k_{\text{reset}} \in \{0, 1\}$ has been introduced:

$$\dot{\tilde{\beta}} = -\frac{\dot{v}}{v_x} \sin \tilde{\beta} + \frac{a_y}{v_x} - \dot{\psi} - k_{\text{reset}} \tilde{\beta}. \tag{35}$$

Kalman Filter

The reset k_{reset} is enabled ($k_{\text{reset}} = 1$) when:

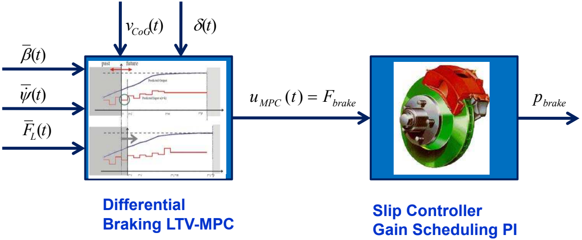
1. the steering wheel is in neutral position: $|\delta| < \Delta_\delta$;
2. the vehicle is not rotating: $|\dot{\psi}| < \Delta_{\dot{\psi}}$;
3. the estimated value of β is not equal to zero $|\tilde{\beta}| > \Delta_\beta$;
4. conditions 1), 2), 3), are true for a certain period of time T_{On};

where Δ_δ, $\Delta_{\dot{\psi}}$ and Δ_β are suitable thresholds.

Outline

Controller Scheme

In this section we describe in detail the design of a "controller" of general scheme about lateral vehicle dynamics control strategy presented before which uses only active braking to track the vehicle reference side slip angle and the reference yaw rate.



It is composed by a differential braking Linear Time Varying Model Predictive Control and by a slip controller.

Nonlinear MPC

Consider the following generic discrete-time nonlinear system:

$$\xi\,(t+1) = f\,(\xi(t), u(t))\,, \tag{36}$$

where $f\,(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, with $f\,(\cdot,\cdot) \in \mathcal{C}^1$, is the state update function, $\xi \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control inputs vector, and $f(0,0) = 0$. Let the system (36) be subject to the following states and inputs constraints:

$$\xi(t) \in \mathcal{X} \text{ and } u(t) \in \mathcal{U} \tag{37}$$

where $\mathcal{X} \in \mathbb{R}^n$ and $\mathcal{U} \in \mathbb{R}^n$ are polytopes.
The control objective is to steer the state of the system (36) to the origin $x_e = 0$, $u_e = 0$.

Nonlinear MPC

Given $N \in Z^{+}$, we consider the cost function $J_N(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^{Nm} \longrightarrow \mathbb{R}^+$ defined as follows:

$$J_N(\xi(t), U(t)) = \Sigma_{k=t}^{t+N-1} l\left(\xi(k), u(k)\right) + P\left(\xi(t+N)\right) \tag{38}$$

where $U(t) = [u(t), ..., u(t+N-1)]$ is a sequence of inputs over the time horizon N; $\xi(k)$ for $k = t, ..., t+N$ is the state trajectory obtained by applying the control sequence $U(t)$ to the system (36), starting from the initial state $\xi(t)$; $l\left(\cdot,\cdot\right) : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^+$, with $l(\cdot,\cdot) \in \mathcal{C}^1$, is the stage cost and $P\left(\cdot\right) : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is the terminal cost.

Nonlinear MPC

At each sampling time t we assume that a state measurement $\xi(t)$ is available and solve the following optimization problem:

$$\min_{U_t} J_N\left(\xi(t), U_t\right) \tag{39a}$$

$$\text{subject to } \xi_{k+1,t} = f(\xi_{k,t}, u_{k,t}) \tag{39b}$$

$$k = t, ..., N-1$$

$$\xi_{k,t} \in \mathcal{X} \quad k = t+1, ..., t+N-1 \tag{39c}$$

$$u_{k,t} \in \mathcal{U} \quad k = t, ..., t+N-1 \tag{39d}$$

$$\xi_{t,t} = \xi\left(t\right) \tag{39e}$$

$$\xi_{N,t} \in \mathcal{X}_f \tag{39f}$$

where (39f) is a final state constraint and $\mathcal{X}_f$ is a polytope.

Nonlinear MPC

Denote by $U_t^* = [u_{t,t}^*, u_{t+1,t}^*, ..., u_{t+N-1,t}^*]$ the optimal solution of (39) at time t; by $\xi_{k,t}^*$ for $k = t+1, .., t+N$ the optimal state trajectory obtained by applying the optimal input sequence U_t^* to the system (39b); and by $J_N^*(\cdot)$ the value function of (39) at time t. The first sample of U_t^* is applied to the plant:

$$u\left(\xi(t)\right) = u_{t,t}^* \tag{40}$$

and, at the next the sampling time, the optimization problem (39) is solved over a shifted horizon.

The problem (39) is a nonlinear and in general non convex optimization problem with N(n + m) optimization variables, nN nonlinear equality constraints (constraints (39b)), and a number of linear constraints (constraints (39c)–(39d)) depending on the polytopes $\mathcal{X}$ and $\mathcal{U}$. The control law (39)–(40) is referred to as Non-Linear Model Predictive Control (NLMPC).

Nonlinear MPC

Remark

In the formulation of the constrained finite-time optimal control (CFTOC) problem (39), *we distinguish between the state* $\xi(t)$ *and input control* $u(t)$ *of the systems* (36) *and the variables* $\xi_{k,t}$ *and* $u_{k,t}$ *of the optimization problem* (39).

The problem (39) is solved by means of nonlinear optimization solvers. The computational burden for solving (39) in general depends on *i)* the order of the system (36), *ii)* the horizon length N, *iii)* the nonlinearities in the function $f(\xi; u)$ and *iv)* the optimization method employed.

Outline

Linear Time Varying (LTV) MPC

We formulate a Linear Time Varying (LTV) model predictive control problem in order to achieve the lateral vehicle dynamics control.
It is a suboptimal MPC algorithm with a lower computational complexity, compared to the NMPC law (39)–(40), which is obtained by approximating the system (36) with a linear time-varying model.
Consider the state $\xi_0 \in \mathcal{X}$ and the input $u_0 \in \mathcal{U}$. Denote by $\hat{\xi}_0(k)$ for $k \geq 0$ the state trajectory obtained by applying the input sequence $u(k) = u_0$ for $k \geq 0$ to the system (36) with $\hat{\xi}_0(0) = \xi_0$:

$$\hat{\xi}_0(k+1) = f\left(\hat{\xi}_0(k), u(k)\right), \tag{41a}$$

$$u(k) = u_0 \tag{41b}$$

$$\hat{\xi}_0(0) = \xi_0 \tag{41c}$$

Linear Time Varying (LTV) MPC

System (36) can be approximated by the following LTV system:

$$\Delta\xi\left(k+1\right) = A_{k,0}\Delta\xi\left(k\right) + B_{k,0}\Delta u\left(k\right) + d_{k,0} \tag{42}$$

where $A_{k,0} \in \mathbb{R}^{n\times n}$ and $B_{k,0} \in \mathbb{R}^{n\times m}$ are defined as

$$A_{k,0} = \left.\frac{\partial f}{\partial \xi}\right|_{\hat{\xi}_0(k),u_0} \qquad B_{k,0} = \left.\frac{\partial f}{\partial u}\right|_{\hat{\xi}_0(k),u_0} \tag{43a}$$

$$d_{k,0} = f(\hat{\xi}_0(k), u_0) \tag{43b}$$

$$\Delta\xi\left(k\right) = \xi\left(k\right) - \hat{\xi}_0(k) \qquad \Delta u\left(k\right) = u\left(k\right) - u_0 \tag{43c}$$

The LTV system (43a) describes the deviations of the nonlinear system (36) from the state trajectory $\hat{\xi}_0(k)$, when a constant sequence of amplitude u_0 is applied.

Remark

System (42) *is a first order approximations of the system* (36) *around the nominal state trajectory* $\hat{\xi}_0(k)$, $k \geq 0$.

Linear Time Varying (LTV) MPC

At each sampling time t we consider the cost function $J_N(\xi(t), U(t))$ in (38). We assume that a state measurement $\xi(t)$ is available and solve the following optimization problem

$$\min_{U_t} J_N(\xi(t), U_t) \tag{44a}$$

$$\text{subject to } \xi_{k+1,t} = A_{k,t}\xi_{k,t} + B_{k,t}\delta u_{k,t} + d_{k,t} \tag{44b}$$

$$k = t, ..., N-1$$

$$\xi_{k,t} \in \mathcal{X} \quad k = t+1, ..., t+N-1 \tag{44c}$$

$$u_{k,t} \in \mathcal{U} \quad k = t, ..., N-1 \tag{44d}$$

$$\xi_{t,t} = \xi(t) \tag{44e}$$

$$\xi_{N,t} \in \mathcal{X}_f \tag{44f}$$

Linear Time Varying (LTV) MPC

In (44) $U_t = [u_{t,t}, u_{t+1,t}, ..., u_{t+N-1,t}]$ is the optimization vector at time t, $\xi_{k,t}$ is the predicted state at time k, with $k = t+1, .., t+N$, given the state measurement $\xi(t)$ at time t and obtained by starting from the state $\xi_{t,t} = \xi(t)$ and applying to the system (44b) the input sequence $u_{t,t}, u_{t+1,t}, ..., u_{t+N-1,t}$. The matrix $A_{k,t}$, $B_{k,t}$ and the vector $d_{k,t}$ are defined as in (43), where the fixed index 0 is replaced by t:

$$A_{k,t} = \left.\frac{\partial f}{\partial \xi}\right|_{\hat{\xi}_t(k),u_t} \qquad B_{k,t} = \left.\frac{\partial f}{\partial u}\right|_{\hat{\xi}_t(k),u_t} \tag{45a}$$

$$d_{k,t} = f(\hat{\xi}_t(k), u_t) \tag{45b}$$

and

$$\hat{\xi}_{k+1,t} = f\left(\hat{\xi}_{k,t}, u_t\right) \quad k = t, ..., t+N-1 \tag{46a}$$

$$\hat{\xi}_{t,t} = \xi(t). \tag{46b}$$

Linear Time Varying (LTV) MPC

As in (39), once a solution $U_t^* = [u_{t,t}^*, u_{t+1,t}^*, ..., u_{t+N-1,t}^*]$ of the problem (44) is available, the first sample of U_t^t

$$u\,(\xi(t)) = u_{t,t}^* \tag{47}$$

is applied to the model to (36).

Remark

The cost function (38) is convex piecewise linear or quadratic, the constraints (44b)–(44f) are linear, therefore the optimization problem (44) is convex. It can be solved with efficient Linear Programming (LP) or Quadrating Programming (QP) solvers, if the functions $l(\xi, u)$ and $P(\xi)$ in (38) are linear or quadratic, respectively.

Linear Time Varying (LTV) MPC

Remark

Although the complexity of the optimization problem (44) *greatly reduces compared to the problem* (39), *we point out that the MPC formulation* (44) *requires N linearizations of the model* (36). *This setup time can be significant for high order models and long prediction horizons.*

Remark

In order to further reduce the computational complexity of the MPC scheme (44), *in the following we will assume that* $A_{k,t} = A_t$, $B_{k,t} = B_t$ *for* $k = t, ..., t + N - 1$.