PhD Course on Vehicle Dynamics Control. Yaw Damping via Front and Rear Steering

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July 8, 2013

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Lecture content

1. Motivation
2. Steering dynamics
3. Decoupling of lateral and yaw dynamics
4. Rear steering for yaw damping
5. Extension of the decoupling to arbitrary mass distribution

Note: The following notes have been extracted from

- "Robust yaw damping of cars with front and rear wheel steering", J. Ackermann, W. Sienel. *IEEE Transactions on Control Systems Technology, vol. 1, no. 1, March 1993*

Lecture content

1. Motivation

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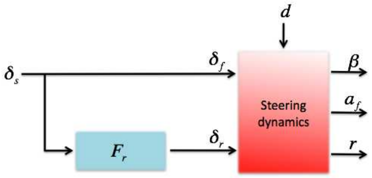

chose the velocity dependent pre-filter as

$$F_r(s, v) = K(v)\frac{1 + T_D(v)s}{1 + T_1(v)s},$$

where $K < 0$ at low speed for better maneuverability in, e.g., parking. At high speed $K > 0$.

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   Advantages

   1. The driver has to focus on the lateral motion only. Disturbances on the yaw dynamics are rejected by the rear-steering.
   2. The yaw damping can be varied without affecting the lateral vehicle behavior

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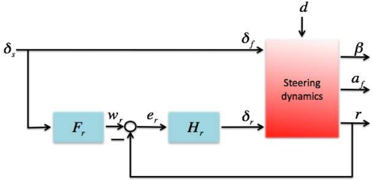

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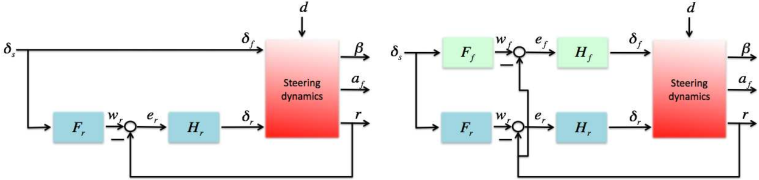

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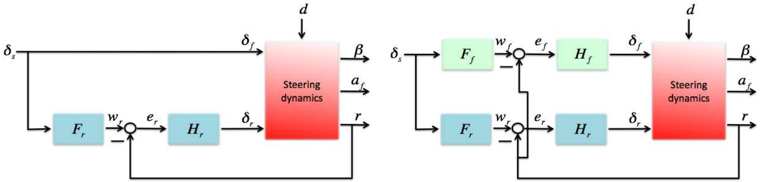

3. Simple control design highlighting the properties of the yaw and lateral response to the front and rear steering

Lecture content

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2. Steering dynamics

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Assumption

The vehicle mass is distributed into two masses concentrated at the front and the rear axle.

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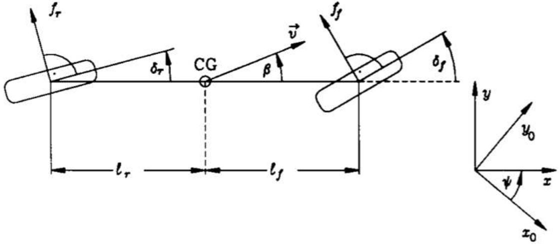

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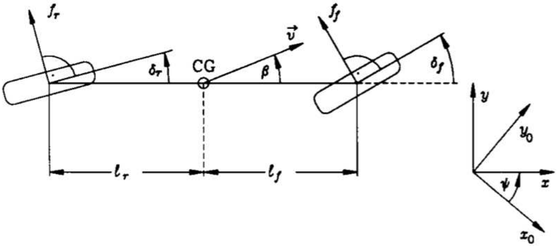
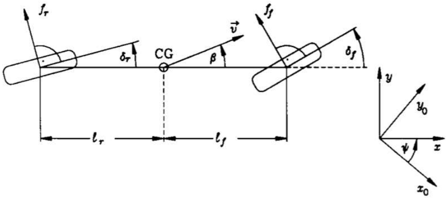


In this case the inertia J and the mass m are related by

$$J = m\ell_r\ell_f$$

Steering dynamics



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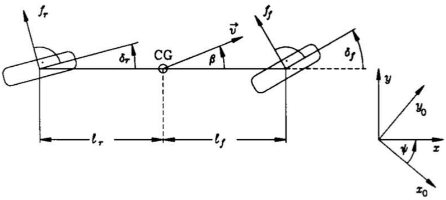

The linearized steering dynamics are given by

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \delta_f \\ \delta_r \end{bmatrix},$$

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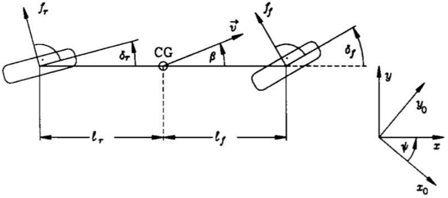

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where

$$\begin{aligned} a_{11} &= -(c_r + c_f)/mv, & a_{12} &= -1 + (c_r\ell_r - c_f\ell_f)/mv^2, \\ a_{21} &= (c_r\ell_r - c_f\ell_f)/m\ell_r\ell_f, & a_{22} &= -(c_r\ell_r^2 + c_f\ell_f^2)/mv\ell_r\ell_f, \end{aligned}$$

$$\begin{aligned} b_{11} &= c_f/mv, & b_{12} &= c_r/mv, \\ b_{21} &= c_f/m\ell_r, & b_{22} &= -c_r/m\ell_f. \end{aligned}$$

Steering dynamics

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$$\begin{aligned} \omega_0^2 &= \frac{c_r c_f \ell^2 + mv^2(c_r\ell_r - c_f\ell_f)}{m^2v^2\ell_r\ell_f}, \\ D_0 &= \frac{\ell(c_r\ell_r + c_f\ell_f)}{2\sqrt{\ell_r\ell_f\left[c_r c_f \ell^2 + mv^2(c_r\ell_r = c_f\ell_f)\right]}} \end{aligned}$$

Lecture content

1. Motivation
2. Steering dynamics
3. Decoupling of lateral and yaw dynamics

Decoupling of lateral and yaw dynamics

Theorem

The steering control law given by $H_f(s) = \frac{1}{s}$ is decoupling. I.e.,

1. r and δ_f are unobservable from a_f,
2. a_f is not controllable from δ_r.

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Proof

1. Augment the state-space model by adding $\dot{\delta}_f = e_f$.

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Proof

1. Augment the state-space model by adding $\dot{\delta}_f = e_f$.
2. Transform the state-space model to the new state vector $[a_f \; r \; \delta_f]$, where

$$\begin{aligned}
a_f &= v(r + \dot{\beta}) + \ell_f \dot{r} \\
&= v(r + a_{11}\beta + a_{12}r + b_{11}\delta_f + b_{12}\delta_r) \\
&+ \ell_f(a_{21}\beta + a_{22}r + b_{21}\delta_f) + b_{22}\delta_r \\
&= d_0(-\beta - \ell_f r/v + \delta_f), \quad d_0 = \ell c_f / m\ell_r
\end{aligned}$$

Decoupling of lateral and yaw dynamics

Proof (Cont.)

3. Close the feedback loop with

$$\dot{\delta}_f = w_f - r$$

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The model resulting from steps 1) - 3) is

$$\begin{bmatrix} \dot{a}_f \\ -- \\ \dot{r} \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} d_{11} & | & 0 & 0 \\ -- & -- & -- & -- \\ d_{21} & | & d_{22} & d_{23} \\ 0 & | & -1 & 0 \end{bmatrix} \begin{bmatrix} a_f \\ -- \\ r \\ \delta_f \end{bmatrix} + \begin{bmatrix} d_0 & 0 \\ -- & -- \\ 0 & d_{22} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_f \\ \delta_r \end{bmatrix},$$

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Decoupling of lateral and yaw dynamics

Proof (Cont.)

where

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\begin{aligned}
d_{11} &= -\ell c_f / m v \ell_r \\
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d_{22} &= -c_r \ell / m v \ell_f \\
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$$

In conclusion

1. r and δ_f can't be observed from a_f,
2. a_f is not controllable from δ_r.

□

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with characteristic polynomial

$$p_I(s) = \omega_I^2 + 2 D_I \omega_I s + s^2,$$

and

$$\omega_I^2 = \frac{c_r}{m\ell_f},\; D_I = \frac{\ell}{2v}\sqrt{\frac{c_r}{m\ell_f}}.$$

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For $v < v_\omega$ the decoupled vehicle has lower natural frequency. Note that v_ω is independent of the speed.

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Theorem

The rear steering control law

$$\delta_r = (\ell/v - k_D)(w_r - r),$$

yields to speed independent eigenvalues of the yaw dynamics.

Decoupling of lateral and yaw dynamics

Proof.

Plug $\delta_r = (\ell/v - k_D)(w_r - r)$ into the decoupled yaw and controller dynamics

$$\begin{aligned} \begin{bmatrix} \dot{r} \\ \dot{\delta}_f \end{bmatrix} &= \begin{bmatrix} d_{22} - (\ell/v - k_D)b_{22} & d_{23} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r \\ \delta_f \end{bmatrix} + \begin{bmatrix} d_{21} \\ 0 \end{bmatrix} a_f \\ &+ \begin{bmatrix} 0 & b_{22}(\ell/v - k_D) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_f \\ w_r \end{bmatrix} \end{aligned}$$

$\square$

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$$\omega_{II}^2 = \frac{c_r}{m\ell_f},\ D_{II} = \frac{k_D}{2}\sqrt{\frac{c_r}{m\ell_f}} = \frac{k_D\omega_{II}}{2}.$$

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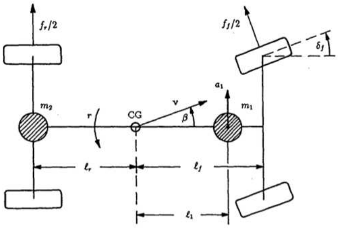

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In case of arbitrary mass distribution

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\begin{aligned}
m &= m_1 + m_2,\ J = m_1\ell_1^2 + m_2\ell_r^2 \\
m_1\ell_1 &= m_2\ell_r,\ J = m_2\ell_1\ell_r + m_1\ell_1\ell_r \\
J &= m\ell_1\ell_r
\end{aligned}
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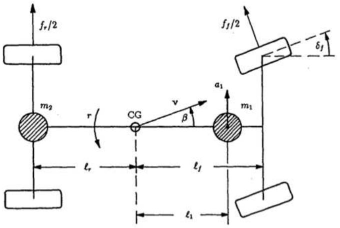

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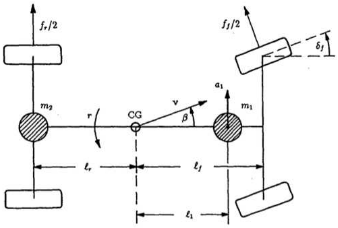
$$m_1 = \frac{m^2\ell_r^2}{J + m\ell_r^2},\ m_2 = \frac{mJ}{J + m\ell_r^2}$$



Decoupling with arbitrary mass distribution

Assume $\dot{v} = 0$ and small δ_f. Write the lateral and yaw dynamics as

$$\begin{bmatrix} mv(\dot{\beta} + r) \\ J\dot{r} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \ell_f & -\ell_r \end{bmatrix} \begin{bmatrix} f_f(\alpha_f) \\ f_r(\alpha_r) \end{bmatrix}.$$

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Recall that

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$$\alpha_r = -\beta_r \qquad \alpha_r = \delta_r - \beta_r$$

Introduce now $\beta_1 = \beta + \frac{\ell_1 r}{v}$ as state variable

$$\begin{bmatrix} mv(\dot{\beta}_1 + r - \frac{\ell_1 \dot{r}}{v}) \\ J\dot{r} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \ell_f & -\ell_r \end{bmatrix} \begin{bmatrix} f_f(\alpha_f) \\ f_r(\alpha_r) \end{bmatrix},$$

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$$\begin{bmatrix} mv(\dot{\beta}_1 + r - \frac{\ell_1 \dot{r}}{v}) \\ J\dot{r} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \ell_f & -\ell_r \end{bmatrix} \begin{bmatrix} f_f(\alpha_f) \\ f_r(\alpha_r) \end{bmatrix},$$

with

$$\alpha_f = \delta_f - \beta_1 - r\frac{\ell_f - \ell_1}{v}, \quad \alpha_r = -\beta_1 + r\frac{\ell_r + \ell_1}{v}$$

Decoupling with arbitrary mass distribution

The dynamics of the yaw and the side slip of mass m_1 can be rearranged as

$$\begin{bmatrix} \dot{\beta}_1 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \ell/mv\ell_r & 0 \\ \ell_f/J & -\ell_r/J \end{bmatrix} \begin{bmatrix} f_f(\alpha_f) \\ f_r(\alpha_r) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

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Observe that,

1. the rear tire force f_r has no direct effect on β_1

Theorem

The front steering control law

$$\dot{\delta}_f = w_f - r + \dot{r}\frac{\ell_f - \ell_1}{v},$$

renders the yaw rate r unobservable from the from tire slip angle α_f.

Decoupling with arbitrary mass distribution

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1. the rear tire force f_r has no direct effect on β_1
2. f_r has an indirect effect on β_1 through r

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Decoupling with arbitrary mass distribution

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Observe that,

1. the rear tire force f_r has no direct effect on β_1
2. f_r has an indirect effect on β_1 through r
3. r enters linearly in the β_1 dynamics

Theorem

The front steering control law

$$\dot{\delta}_f = w_f - r + \dot{r}\frac{\ell_f - \ell_1}{v},$$

renders the yaw rate r unobservable from the from tire slip angle α_f.

Decoupling with arbitrary mass distribution

Proof.

Recall that $\alpha_f = \delta_f - \beta_1 - r\frac{\ell_f - \ell_1}{v}$.

$\square$

Decoupling with arbitrary mass distribution

Proof.

Recall that $\alpha_f = \delta_f - \beta_1 - r\frac{\ell_f - \ell_1}{v}$. Differentiate α_f

$$\begin{aligned}\dot{\alpha}_f &= \dot{\delta}_f - \dot{\beta}_1 - \dot{r}\frac{\ell_f - \ell_1}{v} \\ &= \dot{\delta}_f - \frac{\ell}{mv\ell_r} f_f(\alpha_f) + r - \dot{r}\frac{\ell_f - \ell_1}{v}\end{aligned}$$

$\square$

Decoupling with arbitrary mass distribution

Proof.

Recall that $\alpha_f = \delta_f - \beta_1 - r\dfrac{\ell_f - \ell_1}{v}$. Differentiate α_f

$$\begin{aligned}\dot{\alpha}_f &= \dot{\delta}_f - \dot{\beta}_1 - \dot{r}\frac{\ell_f - \ell_1}{v}\\ &= \dot{\delta}_f - \frac{\ell}{mv\ell_r} f_f(\alpha_f) + r - \dot{r}\frac{\ell_f - \ell_1}{v}\end{aligned}$$

Plug $\dot{\delta}_f = w_f - r + \dot{r}\dfrac{\ell_f - \ell_1}{v}$ in the expression of $\dot{\alpha}_f$

$$\dot{\alpha}_f = w_f - \frac{\ell}{mv\ell_r} f_f(\alpha_f)$$

$\square$

Decoupling with arbitrary mass distribution

Observe that, under the

Assumption

The vehicle mass is distributed into two masses concentrated at the front and the rear axle.

$\ell_1 = \ell_f$ and the front steering control law reduces to the previous.